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M.A. Rahmatulin : A theory of double velocity for the boundary layer in a homogeneous liquid with two components.

By the work of Prandtl and his school the theory of boundary layers has become one of the universal methods in the mechanics of the continua during the last decades. This theory, however, to-day is applied for solving several questions which are not directly connected with the problem of motion of viscous fluids.
In the SSSR the boundary layer theory has been applied during the last time for investigations of the following kind :

1) Calculation of high-powered, nonpunctiform explosions (G.G. Černij).
2) Calculation of instationary filter processes (P.Ja. Kočin, G. Barenblat).
3) Calculation of instationary heat transfer for a heat-transfer coefficient depending on the temperature (Ju.N. Dem'janov).

The aim of this report is to call the researcher's attention still to another field of application of the boundary layer theory. Here I mainly think of the application of the boundary layer theory to the so-called "double-velocity hydro- and aerodynamics" according to Landau's terminology. In order to clear the problem I want to give a survey on some concrete tasks of double-velocity hydro- and aerodynamics which were formulated corresponding to the boundary layer problems :

1) The flow of a homogeneous fluid over a porous surface. In this case the task can be formulated in such a way that a part of the ideal fluid with the mean density ϱ_1 and the speed $\bar{V}_1$ flows

through the surface, while another part with the mean density ϱ_2 and the speed $\bar{V}_2$ flows along. If it is put $\varrho = \varrho_1 + \varrho_2$ as real density, then the boundary layer conditions are evidently :

(1.1) $k\,\bar{V}_{1n} = P$

(1.2) $\bar{V}_{2n} = 0$

furthermore for $x^2 + y^2 + z^2 \longrightarrow \infty$ it holds :

(1.3) $\bar{V}_1 \rightarrow \bar{V}_2 \rightarrow \bar{V}_o$

Although such a double-velocity state is everywhere theoretically possible, it really occurs in that range only which lies directly on the porous wall. For ideal or viscous fluids the boundary layer near the wall must be investigated. This is, however, not the classical boundary layer problem for which the flow is given in the layer, but a case in which the boundary layer and the external flow mutually influence each other - similar to the case of high supersonic velocities. Thereby the flow around a porous plate in supersonic must be solved as the first task.

2) The problem of porous cooling. In its character this problem does not differ from the first one; only the condition (1.1) must be replaced by a corresponding other one.

3) The problem of the boundary-layer control. After the things mentioned above it is not necessary to explain this in detail.

4) Furthermore the double-velocity hydrodynamics is of importance for the motion of such liquids and gases which are saturated with macroscopic dust particles. Here the question is essentially a generalization of the filter theory.

The last case shall be discussed somewhat more detailed.

The deduction of the differential equations for the boundary layer in a liquid with double velocities :

I think that we can consider with good reason liquids and gases saturated with macroscopic dust particles as a mixture of several continuous media. (Footnote : Prof. Voronec, Belgrad called my attention to the possibility to consider the mentioned media as continuous. This is most suitable also in the sense of a verdict of Zukovskij : "Mechanics is the art to set up integrable differentia equations".)

It is evident that the tension and deformation relations are valid for each of the media taking part in the motion. (Footnote: see H.A. Rahmatulin : "Foundations of gas dynamics of mutually penetrating motions of compressible media", Priklad.Mat.Mech.20, 1956. A summary of the results of this paper was published in the reports of the Congress for Theoretical and Applied Mechanics in Brussels.

It holds :

$$\sigma_{ix} = P + 2\mu_i \frac{\partial u_i}{\partial x} - \frac{2}{3}\mu_i \operatorname{div} \overline{V}_i$$

(2.1) correspondingly σ_{iy} and σ_{iz} ,

$$\tau_{ixy} = \mu_i\left(\frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial x}\right)$$

correspondingly τ_{iyz} and τ_{ixz} .

Under the penetration of two media the index i runs from 1 to 2. We introduce the expressions :

$$f_1 = \frac{\varrho_1}{\varrho_{1w}} \quad ; \quad f_2 = \frac{\varrho_2}{\varrho_{2w}}$$

where ϱ_1 and ϱ_2 are the mean densities and ϱ_{1w} and ϱ_{2w}

the real ones of the two media.

The following equations of motion are obtained with the aid of these expressions :

$$\varrho_i \frac{du_i}{dt} = - f_i \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} f_i \mu_i \left(2 \frac{\partial u_i}{\partial x} - \frac{2}{3} \operatorname{div} \bar{V}_i\right)$$

$$+ \frac{\partial}{\partial y} f_i \mu_i \left(\frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial x}\right) + K(u_2 - u_1)$$

(2.2) (correspondingly also an equation with v_i)

$$\frac{\partial(\varrho_i u_i)}{\partial x} + \frac{\partial(\varrho_i v_i)}{\partial y} = 0$$

$$\sum f_i = 0 \quad ; \quad i = 1,2 \quad .$$

These equations contain the following unknowns :

$$u_1 , \quad u_2 , \quad v_1 , \quad v_2 , \quad \varrho_1 , \quad \varrho_2 , \quad P \quad .$$

If the densities ϱ_1 and ϱ_2 are assumed to be constant in the equations of motion so that the condition $f_1 + f_2 = 1$ is satisfied, then six equations with five unknowns remain over. Therefore the mean densities ϱ_1 and ϱ_2 are not allowed to be assumed as constant in a double-velocity fluid, each component of which is incompressible. Nevertheless this assumption for the motions in the boundary layer does not lead to a redundancy in determination of the problem, since in this case two equations drop out. Namely, if it is put ϱ_1 = const and ϱ_2 = const and, if it is estimated for the motions in the boundary layer, then it remains :

$$\rho_1 \frac{du_1}{dt} = - f_1 \frac{\partial P}{\partial x} + f_1 \mu_1 \frac{\partial^2 u_1}{\partial y^2} + K(u_2 - u_1)$$

$$\rho_2 \frac{du_2}{dt} = - f_2 \frac{\partial P}{\partial x} + f_2 \mu_2 \frac{\partial^2 u_2}{\partial y^2} + K(u_1 - u_2)$$

(2.3)

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \ .$$

Thus one obtains four equations for four unknowns.

If the pressure gradient vanishes, then the integral relations of v.Karman can be applied and one obtains according to Pohlhausen :

$$u_1 = U\left[\frac{3}{2}\frac{y}{\delta_1} - \frac{1}{2}\left(\frac{y}{\delta_2}\right)^3\right] \qquad \text{for} \quad 0 \leqslant y \leqslant 1$$

$$u_2 = U\left[\frac{3}{2}\frac{y}{\delta_2} - \frac{1}{2}\left(\frac{y}{\delta_2}\right)^3\right] \qquad \text{for} \quad 0 \leqslant y \leqslant 2$$

$$u_2 = U \qquad\qquad \text{for} \quad \delta_2 \leqslant y \leqslant \delta_1$$

for the thickesses of the boundary layer we obtain :

(2.4)

$$\frac{d\delta_1}{dx} = \frac{\alpha_1}{\delta_1} - \beta_1(\delta_1 - \delta_2)$$

$$\frac{d\delta_2}{dx} = \frac{\alpha_2}{\delta_2} + \beta_2\delta_2(5-6\lambda + \lambda^3) \qquad \text{with} \quad \lambda = \frac{\delta_2}{\delta_1} \ .$$

It can be shown that for $\alpha_2 < \alpha_1$ it always holds $\delta_2 < \delta_1$. If x is eliminated from the equations (2.4), then it follows :

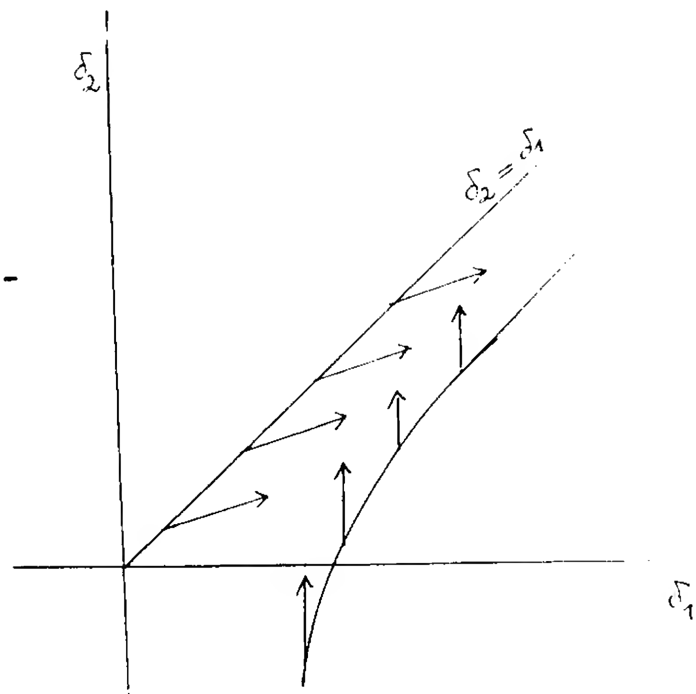
$$(2.5)\quad \frac{d\delta_2}{d\delta_1} = \frac{\alpha_2/\delta_2 + \beta_2\delta_2(5 - 6\lambda + \lambda^3)}{\alpha_1/\delta_1 - \beta_1(\delta_1 - \delta_2)}$$

The isoclinics of this equation are :

$$\delta_2 = \delta_1 \quad \text{with} \quad \frac{d\delta_2}{d\delta_1} = \frac{\alpha_2}{\alpha_1} < 1 \quad ,$$

$$\delta_2 = \delta_1 - \frac{\alpha_1}{\beta_1\delta_1} = f(\delta_1) \quad \text{with} \quad \frac{d\delta_2}{d\delta_1} = \infty \quad .$$

Figure 1 shows a picture of the isoclines. From this it can be seen that the curve $\delta_2 = \delta_2(\delta_1)$ lies below a straight line. A detailed calculation shows that δ_2 very quickly tends to δ_1 and both tend to a value δ which corresponds to the boundary-layer thickness of a liquid with the density $\varrho_1 + \varrho_2$.



If a pressure gradient exists, then u_1 and u_2 and consequently also δ_1 and δ_2 are different from each other in every arbitrary point. Therefore a double-velocity theory in this case surely gives results which differ from the corresponding results in a one-component medium.

A.A. Nikol'skij : On some exact potential flows with separation and their treatment from the aspect of boundary-layer theory.

In his former paper on "The origin of vortices in ideal fluids" Ludwig Prandtl directed to the possibility and suitability of investigating the origin and further development of spiral surfaces of discontinuity on shelves, particularly the similar flows of this kind.

Investigations of this form of motion which I shall denote as "second form of motion", have become particularly necessary during the last time, since the forces acting on thin wings of small aspect ratio for an afflux in fluids or gases under nonvanishing angle of incidence are mainly determined by the second form of motion. Furthermore the flow around obstacles by impact waves even depends on the second form of motion.

If the body with borders or edges which is assumed to be two-dimensional rests in the moment $t = 0$ and moves for $t > 0$ according to the law $v = v_0(t)$, then the second form of motion develops on the edges for

$$t \ll \frac{1}{L} \int_0^t v_0(t)\, dt \quad .$$

Here L is a length which is characteristic for the body. The flow of the second form of motion has a complex potential, the principal term of which is in the x,y - coordinate system

$$W = \varphi + i\psi = c\, L^{1-n}\, v_0(t)\, z^n \quad .$$

Here is

$$n = \frac{\pi}{2\pi - \theta}$$

with the edge angle θ_0, c is a nondimensional constant and for z it holds : $z = x + ij$.
We consider the complex initial potential of a continuous flow around an infinite edge of the general form :

$$W_1 = f_1(t)\, z^n + f_2(t)\, z^{2n} \quad .$$

If it is in particular

$$W_1 = k_1 t^m z^n + k_2 t^{\left[2m(1-n)-n\right]/(2-n)} z^{2n}$$

then the flow developing for $t > 0$ is similar to the second form of motion. The general circulation Γ_1 of the vortex varies according to the law

$$\Gamma_1 \sim t^{2m+n} \quad .$$

Here I want to consider the special case of a degenerated flow which I mentioned, in my report at the 9-th International Congress for Theoretical and Applied Mechanics, but which I did not analyze in detail.
If $2m + n = 0$, then Γ_1 does not depend on t. For $2m + n \rightarrow 0$ the vortex spiral contracts to a point. Now for this limit case a simple, exact solution of the problem can be found, if for $m = -n/2$ a discreet vortex is assumed and if it is required that the velocity on the apex of the edge remains finite. If furthermore absence of exterior forces in the vortex point is assumed, then we have :

$$r_1 = k_1^{1/(2-n)} R_1(n,\beta)\sqrt{t} \quad ; \quad \theta_1 = \theta_1(n,\beta)$$

$$\Gamma_1 = k_1^{1/(2-n)} \cos(n,\beta) \quad .$$

Here

$$ß = \frac{k_2^{(2-n)/(1-n)}}{k_1}$$

is a nondimensional parameter.

For the case of a semiinfinite plate it holds :

$$W_1 = k_1 \; t^{-1/4} \sqrt{z} + k_2 \; t^{-1/2} \; z \quad .$$

If one considers the motion in the plane of the nondimensional variables

$$\xi = k_1^{1/(n-2)} \; t^{-1/2} \; x$$

$$\eta = k_1^{1/(n-2)} \; t^{-1/2} \; y \quad ,$$

then the picture point of the considered particle of the x,y-plane moves in the ξ, η-plane on the track curve of a similar motion. The whole ξ, η-plane is uniquely covered with these track curves. For $k_2 = 0$ these trajectories have the form shown in the figure. Each track curve terminates in the point zero and approaches this point with increasing time along a spiral. There exists a track curve γ , starting form the edge itself and winding up to the vortex point. For $2m+n \rightarrow 0$ this track blurs to a spiral vortex form.

If $ß \neq 0$ (and especially if $ß \rightarrow \infty$), then the lateral afflux blows away that region in which the second form of motion developed. This region concentrates itself in the ξ, η-plane around a certain point of the plate which does not coincide with its edge. The tracks of the similar motion then have the form shown in the figure. The separation of flow does not take place in the point A of the plate. The separation point C

from the body lies on the upper side of the plate. There are some values β_{crit} of β, so that for $\beta > \beta_{crit}$ the separation point removes from the point A .

It is suitable to extend the considered limit solution to the case of a viscous fluid. In the equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

the kinematic viscosity ν has the dimension of a circulation, for the dimensional relations :

$$[\nu] = \frac{L^2}{T} \; ; \quad [k_1] = \frac{L^{2-n}}{T^{(2-n)/2}} \quad ; \qquad [\nu] = \left[k_1^{\,2/(2-n)}\right]$$

are valid. Therefore a corresponding set up can be made for the flow of viscous fluids as for the flow of ideal fluids. Here it appears that the well-known solution becomes similar for the dissipation of a discreet vortex with the circulation Γ in the viscous fluid because of the equality of the dimensions $[\Gamma] = [\nu]$.

With the nondimensional magnitudes

$$\xi = k_1^{\,1/(n-2)}\, t^{-1/2}\, x$$

$$\eta = k_1^{\,1/(n-2)}\, t^{-1/2}\, y$$

$$U(\xi, \eta, \zeta) = k_1^{\,1/(n-2)}\, t^{1/2}\, u$$

$$V(\xi, \eta, \zeta) = k_1^{\,1/(n-2)}\, t^{1/2}\, v$$

$$P = k_1^{\,2/(n-2)}\, t\, \frac{p}{\rho}$$

$$\varepsilon = \frac{\nu}{k^{2/(n-2)}} = \text{const}$$

the equation of motion takes the following form :

$$\frac{1}{2}U + (U - \frac{1}{2}\xi)\frac{\partial U}{\partial \xi} + (V - \frac{1}{2}\eta)\frac{\partial V}{\partial \eta} = -\frac{\partial P}{\partial \xi} = \varepsilon(\frac{\partial^2 U}{\partial \xi^2} + \frac{^2V}{\partial \xi^2})$$

$$\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi} = 0 .$$

For smaller values of ε the solution of the problem leads to the calculation of the boundary layer, on the outer surface of which the boundary conditions must be substituted, which result from the corresponding solution for the ideal fluid.

In the case of the plate the problem is clear : Here the boundary layer consists of a layer lying against the plate and of the free layer which is formed as a result of the flow off of the boundary layer from the plate.

In the viscous fluid the discreet vortices are replaced by a continuous vortex distribution, in the center of which the flow is approximatively equal to the flow obtained under dissipation of a single vortex; here difficulties arise in the investigation of the flow around an edge with $\theta_0 < \pi$ as well as for the flow around a plate with $\beta > \beta_{crit}$, since in these two cases the velocity of the fluid particles flowing away from the body becomes equal to zero in the branch point of the flow.